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More on the explicit solutions for a second-order nonlinear boundary value problem

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Abstract

The explicit solutions to the boundary value problem

$$
\begin{aligned}
&x''(t) = \lambda(t)e^{\mu(t)x(t)}\\
&x(0) = x(1) = 0,
\end{aligned}
$$

where λ and μ are continuous functions, are discussed.

Keywords: Explicit solutions; Second-order nonlinear equations; Two-point boundary value problem

1. Introduction

In [1], Agarwal and Loi considered the nth-order nonlinear differential equation

$$
x^{(n)}(t) = f\left(t, x(t), x'(t), \ldots, x^{(n-1)}(t)\right) \tag{1.1}
$$

with multipoint boundary conditions and prove existence and uniqueness results by using Picard's iterative method. As an example, the authors considered the two-point boundary value problem

$$
\begin{aligned}
&x''(t) = \lambda e^{\mu x(t)}\\
&x(0) = x(1) = 0,
\end{aligned}
$$

where λ and μ are constants. They obtained explicit solutions and discussed other qualitative properties of solutions. It is also noted that such a problem occurs in diffusion theory [2].

In [3], Goyal considered a general problem

$$
\begin{aligned}
&x''(t) = \lambda(t)e^{\mu x(t)}\\
&x(0) = x(1) = 0
\end{aligned}
$$

and obtained explicit solutions for when $\mu > 0$ is a constant and $\lambda(t) > 0$ satisfies $\frac{d^2}{dt^2}(\ln \lambda(t)) = 0$.

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Very recently, Bougoffa [4] considered a more general problem

$$x''(t) = \lambda(t)e^{\mu(t)x(t)} \tag{1.2}$$

$$x(0) = x(1) = 0 \tag{1.3}$$

and claimed the following theorem.

Theorem 1 (*[4]*). *Let $\mu(t) > 0$ be a continuously differentiable function such that $\frac{d^2}{dt^2}\left(\frac{1}{\mu(t)}\right) = 0$ and let $\lambda(t) = b\mu^3(t)$, where $b > 0$ is a constant. Then the solutions of (1.2) and (1.3) are given by*

$$x(t) = \frac{1}{\mu(t)} \ln z(t), \tag{1.4}$$

where $z(t)$ is given by

$$z(t) = \frac{c}{2b}\left[\tanh^2\left(\mp\frac{\sqrt{c}}{2}\int \mu^2(t)dt + \frac{d}{2\sqrt{c}}\right) - 1\right] \tag{1.5}$$

in which the constants of integration c and d are obtained from

$$c = \frac{1}{\mu_0^2}\ln^2\left(\frac{1+\sqrt{2n+1}}{\sqrt{2n+1}-1}\right), \qquad d = \frac{(2n+1)\pi}{\mu_0}\ln\left(\frac{1+\sqrt{2n+1}}{\sqrt{2n+1}-1}\right)i \tag{1.6}$$

and

$$c = \frac{1}{\mu_1^2}\ln^2\left(\frac{1+\sqrt{2n+1}}{\sqrt{2n+1}-1}\right), \qquad d = \frac{(2n+1)\pi}{\mu_1}\ln\left(\frac{1+\sqrt{2n+1}}{\sqrt{2n+1}-1}\right)i, \tag{1.7}$$

where

$$\mu_0 = \left(\mp\int \mu^2(t)dt\right)\Big|_{t=0}, \qquad \mu_1 = \left(\mp\int \mu^2(t)dt\right)\Big|_{t=1}, \qquad n = 1, 2, \ldots. \tag{1.8}$$

Unfortunately, the conditions of Theorem 1 conflict each other. To see this, we note that $x(0) = x(1) = 0$ imply that $\mu_0 = \mu_1$ which together with $\frac{d^2}{dt^2}\left(\frac{1}{\mu(t)}\right) = 0$ result in $\mu(t) = 0$. This is a contradiction.

In this work, our aim is to consider the same problem (1.2) and (1.3) and derive conditions for the existence of nontrivial solutions. We should note that our method is similar, except that we have modified the transformation used in [4].

2. Main results

Let

$$\mu(t)x = u + \mu(t)(At + B), \tag{2.1}$$

where A and B are suitable constants such that $A^2 + B^2 \neq 0$. When $A = B = 0$, this substitution coincides with the one in [4]. Substituting this into Eq. (1.2) we get

$$\frac{d}{dt}\left(p(t)\frac{du}{dt}\right) + q(t)\frac{du}{dt} + r(t)u = \lambda(t)e^{\mu(t)(At+B)}e^u, \tag{2.2}$$

where $p(t) = \frac{1}{\mu(t)}$, $q(t) = \frac{d}{dt}\left(\frac{1}{\mu(t)}\right)$ and $r(t) = \frac{d^2}{dt^2}\left(\frac{1}{\mu(t)}\right)$. Multiplying both sides of (2.2) by $\xi(t) = e^{\int \frac{q(t)}{p(t)}dt}$ and taking into account $\xi'(t)p(t) = \xi(t)q(t)$, we obtain

$$\frac{d}{dt}\left(\xi(t)p(t)\frac{du}{dt}\right) + \xi(t)r(t)u = \xi(t)\lambda(t)e^{\mu(t)(At+B)}e^u. \tag{2.3}$$

Now, if $u = \ln z$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\xi(t)p(t)\frac{\mathrm{d}}{\mathrm{d}t}(\ln z)\right) + \xi(t)r(t)\ln z = \xi(t)\lambda(t)\mathrm{e}^{\mu(t)(At+B)}z. \tag{2.4}$$

With the substitution of $v = \frac{z'}{z}$ into Eq. (2.4) and taking into account $z(t) \neq 0$ Eq. (2.4) can be written in the following differential form:

$$vd\,(\xi(t)p(t)v) + \xi(t)r(t)\left(\frac{\ln z}{z}\right)\mathrm{d}z = \xi(t)\lambda(t)\mathrm{e}^{\mu(t)(At+B)}\mathrm{d}z. \tag{2.5}$$

To integrate (2.5), set $w = \xi p v$ and choose

$$\xi^2(t)p(t)r(t) = a \quad \text{and} \quad \xi^2(t)\lambda(t)p(t)\mathrm{e}^{\mu(t)(At+B)} = b > 0,$$

where a and b are constants, that is

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{1}{\mu(t)}\right) = a\mu^3(t) \quad \text{and} \quad \lambda(t) = b\mathrm{e}^{-\mu(t)(At+B)}\mu^3(t). \tag{2.6}$$

We conclude that

$$w^2 + a\ln^2 z = 2bz + c,$$

where c is an arbitrary constant of integration and thus we get, in view of $w = \xi p v$ and $v = \frac{z'}{z}$,

$$\frac{\mathrm{d}z}{z\sqrt{2bz - a\ln^2 z + c}} = \mp\mu^2(t)\mathrm{d}t. \tag{2.7}$$

Unfortunately we are unable to solve z from (2.7) in general. So we also assume that $a = 0$ and $c > 0$ as in [4]. Of course, it would be better to obtain a similar result for $a \neq 0$. This problem remains open.

Integrating both sides of Eq. (2.7) with $a = 0$ and $c > 0$, we obtain

$$\tanh^{-1}\left(\sqrt{\frac{2b}{c}z + 1}\right) = \mp\frac{\sqrt{c}}{2}\int \mu^2(t)\mathrm{d}t + \frac{d}{2\sqrt{c}}, \tag{2.8}$$

where d is an arbitrary constant of integration and straightforward computation yields

$$z(t) = \frac{c}{2b}\left[\tanh^2\left(\mp\frac{\sqrt{c}}{2}\int \mu^2(t)\mathrm{d}t + \frac{d}{2\sqrt{c}}\right) - 1\right] \tag{2.9}$$

or

$$z(t) = -\frac{2c}{b}\frac{\mathrm{e}^{d/\sqrt{c}}\mathrm{e}^{\sqrt{c}(\mp\int\mu^2(t)\mathrm{d}t)}}{\left[1 + \mathrm{e}^{d/\sqrt{c}}\mathrm{e}^{\sqrt{c}(\mp\int\mu^2(t)\mathrm{d}t)}\right]^2}. \tag{2.10}$$

After z has been found then $u = \ln z$ and from (2.1) the general solutions of Eq. (1.2) are given by $x(t) = \frac{1}{\mu(t)}\ln z(t) + At + B$.

In order to find the particular solutions to the given boundary value problem (1.2) and (1.3), we use the boundary conditions (1.3) to find the constants of integration c and d. Indeed from the boundary conditions $x(0) = x(1) = 0$, we have $z(0) = \mathrm{e}^{-B\mu(0)}$ and $z(1) = \mathrm{e}^{-(A+B)\mu(1)}$.

Thus

$$\sqrt{\frac{2b}{c}\mathrm{e}^{-B\mu(0)} + 1} = \tanh\left(\frac{\sqrt{c}}{2}\mu_0 + \frac{d}{2\sqrt{c}}\right) \tag{2.11}$$

and

$$\sqrt{\frac{2b}{c}\mathrm{e}^{-(A+B)\mu(1)}+1}=\tanh\left(\frac{\sqrt{c}}{2}\mu_1+\frac{d}{2\sqrt{c}}\right), \tag{2.12}$$

where μ_0 and μ_1 are as defined before in (1.8).

If we choose $b = nc$ and $n = 1, 2, 3, \dots$ from (2.11) we obtain

$$\mathrm{e}^{\sqrt{c}\mu_0}\mathrm{e}^{d/\sqrt{c}}=\frac{1+\sqrt{2n\mathrm{e}^{-B\mu(0)}+1}}{1-\sqrt{2n\mathrm{e}^{-B\mu(0)}+1}}.$$

Now, if we choose $\mathrm{e}^{d/\sqrt{c}} = -1 = \mathrm{e}^{(2n+1)\pi i}$, we have

$$c=\frac{1}{\mu_0^2}\ln^2\left(\frac{\sqrt{2n\mathrm{e}^{-B\mu(0)}+1}+1}{\sqrt{2n\mathrm{e}^{-B\mu(0)}+1}-1}\right),\qquad d=\frac{(2n+1)\,\pi\mathrm{i}}{\mu_0}\ln\left(\frac{\sqrt{2n\mathrm{e}^{-B\mu(0)}+1}+1}{\sqrt{2n\mathrm{e}^{-B\mu(0)}+1}-1}\right). \tag{2.13}$$

Similarly, with these choices and from (2.12) we have

$$c=\frac{1}{\mu_1^2}\ln^2\left(\frac{\sqrt{2n\mathrm{e}^{-(A+B)\mu(1)}+1}+1}{\sqrt{2n\mathrm{e}^{-(A+B)\mu(1)}+1}-1}\right),\qquad d=\frac{(2n+1)\,\pi\mathrm{i}}{\mu_1}\ln\left(\frac{\sqrt{2n\mathrm{e}^{-(A+B)\mu(1)}+1}+1}{\sqrt{2n\mathrm{e}^{-(A+B)\mu(1)}+1}-1}\right). \tag{2.14}$$

Further to satisfy (2.13) and (2.14), we must have

$$\left(\frac{\sqrt{2n\mathrm{e}^{-B\mu(0)}+1}+1}{\sqrt{2n\mathrm{e}^{-B\mu(0)}+1}-1}\right)^{\mu_1}=\left(\frac{\sqrt{2n\mathrm{e}^{-(A+B)\mu(1)}+1}+1}{\sqrt{2n\mathrm{e}^{-(A+B)\mu(1)}+1}-1}\right)^{\mu_0},\quad n\ge 1. \tag{2.15}$$

Thus, we have proved the following theorem.

Theorem 2. *Let (2.15) hold with $A^2 + B^2 \neq 0$. If $\mu(t) > 0$ is a continuously differentiable function on $t \in [0, 1]$ such that $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{1}{\mu(t)}\right) = 0$ and $\lambda(t) = b\mathrm{e}^{-\mu(t)(At+B)}\mu^3(t)$, where $b > 0$ is a constant, then the solutions of (1.2) and (1.3) are of the form*

$$x(t)=\frac{1}{\mu(t)}\ln z(t)+At+B,$$

where $z(t)$ is given by (2.10) with c and d as in (2.13).

Note that it is possible to obtain the following type of solutions. Indeed, integrating both sides of Eq. (2.7) with $a = 0$ and $c > 0$ and straightforward computation yields

$$z(t)=\frac{c}{2b}\left[\coth^2\left(\mp\frac{\sqrt{c}}{2}\int\mu^2(t)\mathrm{d}t+\frac{d}{2\sqrt{c}}\right)-\frac{c}{2b}\right]$$

or

$$z(t)=\frac{2c}{b}\frac{\mathrm{e}^{d/\sqrt{c}}\mathrm{e}^{\sqrt{c}(\mp\int\mu^2(t)\mathrm{d}t)}}{[1-\mathrm{e}^{d/\sqrt{c}}\mathrm{e}^{\sqrt{c}(\mp\int\mu^2(t)\mathrm{d}t)}]^2}.$$

Using the conditions $z(0) = \mathrm{e}^{-B\mu(0)}$, $z(1) = \mathrm{e}^{-(A+B)\mu(1)}$ and choosing $b = kc$, $k \ge 1$ a real number, we get the values of the constants c and d as follows:

$$c=\frac{1}{(\mu_0^*)^2}\ln^2\left(\frac{\sqrt{2k\mathrm{e}^{-B\mu(0)}+1}+1}{\sqrt{2k\mathrm{e}^{-B\mu(0)}+1}-1}\right),\quad d=0 \tag{2.16}$$

and

$$c=\frac{1}{(\mu_1^*)^2}\ln^2\left(\frac{\sqrt{2k\mathrm{e}^{-(A+B)\mu(1)}+1}+1}{\sqrt{2k\mathrm{e}^{-(A+B)\mu(1)}+1}-1}\right),\quad d=0, \tag{2.17}$$

where $\mu_0^* = \left(\mp \int \mu^2(t)\mathrm{d}t\right)\big|_{t=0}$ and $\mu_1^* = \left(\mp \int \mu^2(t)\mathrm{d}t\right)\big|_{t=1}$ are positive constants. Further to satisfy (2.16) and (2.17), we must have

$$\left(\frac{\sqrt{2k\mathrm{e}^{-B\mu(0)}+1}+1}{\sqrt{2k\mathrm{e}^{-B\mu(0)}+1}-1}\right)^{\mu_1^*} = \left(\frac{\sqrt{2k\mathrm{e}^{-(A+B)\mu(1)}+1}+1}{\sqrt{2k\mathrm{e}^{-(A+B)\mu(1)}+1}-1}\right)^{\mu_0^*}, \quad k \geq 1.$$

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